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Equational properties of iteration in algebraically complete categories

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Abstract

We prove the following completeness theorem: If the fixed point operation over a category is defined by initiality, then the equations satisfied by the fixed point operation are exactly those of iteration theories. Thus, in such categories, the equational axioms of iteration theories provide a sound and complete axiomatization of the equational properties of the fixed point operation.

Keywords: Fixed points; Iteration theories; Algebraically complete categories

1. Introduction

Iteration theories provide an axiomatic treatment of the equational properties of the fixed point (or dagger or iteration) operation in cartesian categories, and in Lawvere theories in particular. The book [11] contains convincing evidence that all of the valid equations which hold for a “constructive” fixed point operation are captured by the axioms of iteration theories. But there are models in which the fixed point operation is not constructive. Let \mathcal{C} be a category with a given collection \mathcal{F} of functors $\mathcal{C}^{n+p} \rightarrow \mathcal{C}^n$, $n, p \geq 0$ containing the projections and closed under composition and target tupling. Suppose that for each $F: \mathcal{C}^{n+p} \rightarrow \mathcal{C}^n$ in \mathcal{F} and each \mathcal{C}^p -object y , there is an initial F_y -algebra $(F^\dagger y, \mu_{F,y})$, where F_y denotes the endofunctor $F(-, y): \mathcal{C}^n \rightarrow \mathcal{C}^n$. It is well-known, see e.g. [11], that the assignment $y \mapsto F^\dagger y$ is the object map of a unique functor $F^\dagger: \mathcal{C}^p \rightarrow \mathcal{C}^n$ such that $\mu_F = (\mu_{F,y})_{y \in \mathcal{C}^p}$ is a natural transformation (in fact isomorphism) $F \cdot \langle F^\dagger, \mathbf{1}_p \rangle \rightarrow F^\dagger$. (Here, $\mathbf{1}_p$ stands for the identity functor $\mathcal{C}^p \rightarrow \mathcal{C}^p$.)

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Borrowing terminology from [22], we call the pair $(\mathcal{C}, \mathcal{F})$ an *algebraically complete category* if F^\dagger is in \mathcal{F} whenever F is. When \mathcal{C} is an ω -category and the functor F preserves colimits of ω -chains, F^\dagger can be constructed by the well-known initial algebra construction, cf. [1, 2, 5, 27, 37, 38], or the book [3]. But unless one has some additional assumptions on the category \mathcal{C} and the functors \mathcal{F} , the dagger operation in algebraically complete categories is not constructive.

In this paper our concern is the logic of initiality. What are all of the equations that hold (up to isomorphism) for the dagger operation in algebraically complete categories? Some questions related to this topic have been studied in several papers, see e.g. [7, 10, 12, 13, 22, 23, 31, 35]. In the case that \mathcal{C} is an ω -category [37, 38] and the functors preserve all ω -colimits, the dagger operation is constructive. Hence the valid equations are exactly those of iteration theories. See [10], or [18] for the case that \mathcal{C} is an ω -cpo. In [12], it has been shown that the dagger operation in algebraically complete categories satisfies *Conway's classical identities* [16] for the regular sets, except for the equation $A^{**} = A^*$. Then in [20], it is shown that in the particular case that the category \mathcal{C} is a poset, so that the initial algebras are least pre-fixed points and hence the Park induction principle holds, the valid equations satisfied by the dagger operation are again those of iteration theories. (Kozen's axiomatization [25] of the regular sets may be seen as an instance of the completeness of the Park induction principle.) In this paper, we generalize this result for the (non-constructive) fixed point operation in algebraically complete categories. This general result seems to indicate that the iteration theory identities also capture the equational properties of non-constructive fixed point operations. Our argument is based on recent advances on the axiomatization of iteration theories: A complete set of axioms consists of a small set of equations and an identity associated with each finite group, see [21].

In programming languages, one may define new data types by taking initial algebras of functors corresponding to data type constructors. (The choice of the right category is a non-trivial task, see [4, 31, 32, 37, 38], to mention only a few references.) Our main result shows that the calculus of iteration theories is a useful formal tool for establishing the equivalence of two specifications. See Section 9.

2. Preliminaries

We refer to [6, 28] for basic notions of categories, and [24, 15] for 2-categories. In any 2-category \mathcal{C} , we will denote horizontal composition by \cdot and vertical composition by \star . Thus, for any horizontal morphisms $f, f' : A \rightarrow B$, $g, g' : B \rightarrow C$, and for any vertical morphisms (or 2-cells) $u : f \rightarrow f'$ and $v : g \rightarrow g'$, both $f \cdot g$ and $f' \cdot g'$ are horizontal morphisms $A \rightarrow C$, and $u \cdot v$ is a vertical morphism $f \cdot g \rightarrow f' \cdot g'$. And if f, g, h are given horizontal morphisms $A \rightarrow B$ and $u : f \rightarrow g$ and $v : g \rightarrow h$, then $u \star v$ is a vertical morphism $f \rightarrow h$. As usual, we write also f for the identity vertical morphism corresponding to a horizontal morphism $f : A \rightarrow B$, and we use the notation 1_A for the horizontal identity $A \rightarrow A$ as well as for the vertical identity $1_A \rightarrow 1_A$.

The two composition operations are related by the *interchange law*:

$$(u \cdot u') \star (v \cdot v') = (u \star v) \cdot (u' \star v'),$$

for all vertical morphisms $u: f \rightarrow g$, $v: g \rightarrow h$, $u': f' \rightarrow g'$ and $v': g' \rightarrow h'$ with $f, g, h: A \rightarrow B$ and $f', g', h': B \rightarrow C$. Moreover, the horizontal composite of two vertical identity morphisms is itself a vertical identity. For any 2-category \mathcal{C} , the 2-cells are the morphisms of a category $\mathbf{Cell}_{\mathcal{C}}$, which is in fact a 2-category with a suitable notion of morphism between 2-cells. See [11] and Section 7 for details.

A 2-theory is a 2-category T whose objects are the natural numbers $n \geq 0$ such that there are *distinguished horizontal morphisms* $i_n: 1 \rightarrow n$, $i \in [n] = \{1, \dots, n\}$ with the following coproduct property: For any 2-cells $\alpha_i = (u_i: f_i \rightarrow g_i): 1 \rightarrow p$, $i \in [n]$, there is a unique 2-cell

$$\alpha: n \rightarrow p$$

such that $i_n \cdot \alpha = \alpha_i$, for all $i \in [n]$. We denote this α as $\langle \alpha_1, \dots, \alpha_n \rangle$. Thus, writing

$$\alpha = (\langle u_1, \dots, u_n \rangle: \langle f_1, \dots, f_n \rangle \rightarrow \langle g_1, \dots, g_n \rangle),$$

we have

$$i_n \cdot \langle f_1, \dots, f_n \rangle = f_i, \quad i_n \cdot \langle g_1, \dots, g_n \rangle = g_i, \quad i_n \cdot \langle u_1, \dots, u_n \rangle = u_i,$$

for all $i \in [n]$. The operation defined by the above coproduct conditions is called *tupling*. In the case that $n=0$, it follows that there is a unique 2-cell $0 \rightarrow n$ determined by a (unique) horizontal morphism 0_n . Further, it follows that each identity 2-cell 1_n is determined by the distinguished morphisms i_n :

$$1_n = \langle 1_n, \dots, 1_n \rangle.$$

As an additional assumption we require that $1_1 = 1_1$, so that $\langle \alpha \rangle = \alpha$, for any 2-cell $\alpha: 1 \rightarrow p$. Note that the underlying category of a 2-theory T determined by the horizontal morphisms is a Lawvere theory, cf. [30]. Any Lawvere theory in turn determines a 2-theory all of whose vertical morphisms are identities.

Morphisms of 2-theories are 2-functors that preserve the distinguished morphisms i_n . It follows that each 2-theory morphism preserves the tupling operation.

Remark 2.1. Several ways of defining limits and colimits in 2-categories are discussed in [15]. Our notion of coproduct corresponds to 2-colimits. Weaker notions include bilimits and lax limits.

Example 2.2. The 2-theory T_0 has horizontal morphisms $n \rightarrow p$ all functions $[n] \rightarrow [p]$. Each horizontal morphism has a vertical identity, and there are no other vertical morphisms. The composite $f \cdot g$ of the horizontal morphisms f and g is their function composite. The 2-theory T_0 is initial in the category of 2-theories.

Example 2.3. Suppose that \mathcal{C} is a category. The 2-theory $\mathbf{Th}(\mathcal{C})$ has horizontal morphisms $n \rightarrow p$ all functors $\mathcal{C}^p \rightarrow \mathcal{C}^n$. (Note the reversal of the arrow.) For $f, g: n \rightarrow p$, a vertical morphism $f \rightarrow g$ is a natural transformation. The definition of horizontal and vertical composition is standard. For each $i \in [n]$, $n \geq 0$, the distinguished morphism i_n is the i th projection $\mathcal{C}^n \rightarrow \mathcal{C}$.

Example 2.4. Milner's synchronization trees [33] and their morphisms form a 2-theory. See [11].

Example 2.5. Suppose that A is a poset. The theory $\mathbf{Th}_m(A)$ has morphisms $n \rightarrow p$ the monotonic functions $A^p \rightarrow A^n$. Since each hom-set is itself partially ordered by the pointwise order, and since the theory operations of composition and tupling preserve the partial order, it follows that $\mathbf{Th}_m(A)$ is a 2-theory. Note that $\mathbf{Th}_m(A)$ has at most one vertical morphism between any two horizontal morphisms $f, g: n \rightarrow p$.

Each 2-theory T contains a least sub 2-theory T'_0 determined by the images of the 2-cells in T_0 under the unique morphism $T_0 \rightarrow T$. The horizontal morphisms $n \rightarrow p$ in T'_0 are those of the form

$$\langle (1\rho)_p, \dots, (n\rho)_p \rangle, \quad (1)$$

where ρ is a function $[n] \rightarrow [p]$. Each vertical morphism in T'_0 is also of this form, since each vertical morphism is a vertical identity. The morphisms (1) are called *base* and are usually identified with the function ρ . (In non-trivial 2-theories, this identification is completely legal, since T'_0 is isomorphic to T_0 .) We call a base morphism surjective, injective, or bijective, if the corresponding function has the appropriate property. A bijective base morphism is sometimes called a *base permutation*.

It follows from the definition that each object $n + m$ of a 2-theory is the coproduct (in the 2-categorical sense) of the objects n and m . Indeed, let κ denote the base morphism corresponding to the inclusion $[n] \rightarrow [n + m]$, and let λ correspond to the translated inclusion $[m] \rightarrow [n + m]$. Then for any 2-cells $\alpha = (u: f \rightarrow f'): n \rightarrow p$ and $\beta = (v: g \rightarrow g'): m \rightarrow p$ there is a unique 2-cell $\langle \alpha, \beta \rangle = (\langle u, v \rangle: \langle f, g \rangle \rightarrow \langle f', g' \rangle): n + m \rightarrow p$ with

$$\kappa \cdot \langle \alpha, \beta \rangle = \alpha, \quad \lambda \cdot \langle \alpha, \beta \rangle = \beta.$$

The operation defined by these conditions is called *pairing* and is an extension of the tupling operation. Another useful operation is that of *separated sum*. Let κ and λ be the base morphisms defined above, and let $\kappa': p \rightarrow p + q$ and $\lambda': q \rightarrow p + q$ be defined in the same way. Then the separated sum of the 2-cells $\alpha = (u: f \rightarrow f'): n \rightarrow p$ and $\beta = (v: g \rightarrow g'): m \rightarrow q$ is the unique 2-cell $\alpha \oplus \beta = (u \oplus v: f \oplus g \rightarrow f' \oplus g'): n + m \rightarrow p + q$ such that

$$\kappa \cdot (\alpha \oplus \beta) = \alpha \cdot \kappa', \quad \lambda \cdot (\alpha \oplus \beta) = \beta \cdot \lambda'.$$

These operations have several useful properties some of which are included below for the reader's convenience.

$$\begin{aligned}\langle \alpha, \langle \beta, \gamma \rangle \rangle &= \langle \langle \alpha, \beta \rangle, \gamma \rangle, & \alpha \oplus (\beta \oplus \gamma) &= (\alpha \oplus \beta) \oplus \gamma, \\ \langle \alpha, 0_p \rangle &= \alpha = \langle 0_p, \alpha \rangle, & \alpha \oplus 0_0 &= \alpha = 0_0 \oplus \alpha, \\ (\alpha \oplus \beta) \cdot \langle \gamma, \delta \rangle &= \langle \alpha \cdot \gamma, \beta \cdot \delta \rangle, & (\alpha \oplus \beta) \cdot \langle \gamma \oplus \delta \rangle &= \alpha \cdot \gamma \oplus \beta \cdot \delta,\end{aligned}$$

whenever the 2-cells $\alpha, \beta, \gamma, \delta$ have appropriate source and target.

Two other useful identities involving vertical composition are:

$$\begin{aligned}\langle u_1 \star v_1, u_2 \star v_2 \rangle &= \langle u_1, u_2 \rangle \star \langle v_1, v_2 \rangle, \\ (u_1 \star v_1) \oplus (u_2 \star v_2) &= (u_1 \oplus u_2) \star (v_1 \oplus v_2),\end{aligned}$$

where the vertical morphisms u_i, v_i , $i = 1, 2$, have appropriate source and target. By these conditions, the interchange law also holds for vertical composition and the pairing operation, or the separated sum operation.

Below we will use the above equations without explicit mention.

3. Algebras in 2-theories

In this section we define f -algebras in a 2-theory.

Definition 3.1. Suppose that $f : n \rightarrow n + p$ is a horizontal morphism in a 2-theory T . An f -algebra (g, u) consists of a horizontal morphism $g : n \rightarrow p$ and a vertical morphism $u : f \cdot \langle g, 1_p \rangle \rightarrow g$. Suppose that (g, u) and (h, v) are f -algebras. An f -algebra morphism $(g, u) \rightarrow (h, v)$ is a vertical morphism $w : g \rightarrow h$ such that

$$u \star w = (f \cdot \langle w, 1_p \rangle) \star v,$$

i.e., the following diagram commutes:

$$\begin{array}{ccc} f \cdot \langle g, 1_p \rangle & \xrightarrow{u} & g \\ f \cdot \langle w, 1_p \rangle \downarrow & & \downarrow w \\ f \cdot \langle h, 1_p \rangle & \xrightarrow{v} & h \end{array}$$

Definition 3.2. Suppose that $f : n \rightarrow n + p$ is a horizontal morphism in the 2-theory T . The f -algebra (g, u) is an *initial f -algebra*, if for each f -algebra (h, v) there exists a unique morphism $(g, u) \rightarrow (h, v)$.

It is well known that if (g, u) is an initial f -algebra then u is a vertical isomorphism.

Definition 3.3. An *algebraically complete 2-theory* is a 2-theory T together with a specified initial f -algebra (f^\dagger, μ_f) , for each horizontal morphism $f : n \rightarrow n + p$.

Example 3.4. Suppose that $(\mathcal{C}, \mathcal{F})$ is an algebraically complete category. Then the functors \mathcal{F} determine a sub 2-theory of $\mathbf{Th}(\mathcal{C})$ that we denote by $\mathbf{Th}(\mathcal{C}, \mathcal{F})$. This 2-theory has all initial f -algebras. Indeed, if $f : n \rightarrow n + p$ in $\mathbf{Th}(\mathcal{C}, \mathcal{F})$, then (f^\dagger, μ_f) is an initial f -algebra, since for each \mathcal{C}^p -object y , $(f^\dagger y, \mu_{f,y})$ is an initial f_y -algebra in the usual sense. See the Introduction. It follows that $\mathbf{Th}(\mathcal{C}, \mathcal{F})$ is an algebraically complete 2-theory.

Example 3.5. Suppose that \mathcal{C} is an ω -category, i.e., \mathcal{C} has an initial object and colimits of all ω -chains. Then \mathcal{C}^n is also an ω -category, for each $n \geq 0$. Further, if \mathcal{F} denotes the collection of all functors $\mathcal{C}^{n+p} \rightarrow \mathcal{C}^n$ which preserve colimits of all ω -chains, $(\mathcal{C}, \mathcal{F})$ is an algebraically complete category. Hence $\mathbf{Th}_\omega(\mathcal{C}) = \mathbf{Th}(\mathcal{C}, \mathcal{F})$ is an algebraically complete 2-theory.

More generally, an ω -continuous 2-theory is a 2-theory T such that each vertical category $T(n, p)$ is an ω -category. Moreover, composition preserves initial objects in the first argument and colimits of ω -chains in either argument. Each ω -continuous 2-theory is algebraically complete.

Example 3.6. In [23], several examples of a category \mathcal{C} are given such that each functor $\mathcal{C} \rightarrow \mathcal{C}$ has an initial algebra. These examples include the category of countable sets and the category of vector spaces of dimension at most countable. If \mathcal{C} is such and \mathcal{F} denotes the collection of *all* functors $\mathcal{C}^{n+p} \rightarrow \mathcal{C}^n$, then $(\mathcal{C}, \mathcal{F})$ is algebraically complete. (The proof of this fact uses the pairing identity, see below.) Thus $\mathbf{Th}(\mathcal{C})$ is also algebraically complete.

Example 3.7. Suppose that A is a poset and T is a subtheory of $\mathbf{Th}_m(A)$, so that T is ordered by the inherited partial order. Suppose that for each $f : n \rightarrow n + p$ in T and $y \in A^p$, there is a least pre-fixed point $f^\dagger y$ of the function $f_y : A^n \rightarrow A^n$, $x \mapsto f(x, y)$. Then the function f^\dagger is also monotonic. If f^\dagger is in T , for each $f : n \rightarrow n + p$ in T , then T is an algebraically complete 2-theory, in fact an algebraically complete category. Several examples of an algebraically complete category with a non-constructive dagger are given in [20].

By Definition 3.3, each algebraically complete 2-theory comes with a dagger operation defined on the horizontal morphisms $n \rightarrow n + p$. In the next section, we will study some equational properties of the dagger operation.

4. Conway theories and iteration theories

In this section we consider some axiomatic classes of *preiteration theories*, i.e., Lawvere theories T (without vertical structure) enriched by a dagger or iteration

operation

$$f : n \rightarrow n + p \mapsto f^\dagger : n \rightarrow p.$$

The *Conway identities* are the following equations:

- LEFT ZERO IDENTITY

$$(0_n \oplus f)^\dagger = f,$$

all $f : n \rightarrow p$. The particular case that $n = 1$ is called the scalar left zero identity.

- RIGHT ZERO IDENTITY

$$(f \oplus 0_q)^\dagger = f^\dagger \oplus 0_q,$$

all $f : n \rightarrow n + p$. The particular case that $n = q = 1$ is called the scalar right zero identity.

- PAIRING IDENTITY

$$\langle f, g \rangle^\dagger = \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, h^\dagger \rangle,$$

for all $f : n \rightarrow n + m + p$ and $g : m \rightarrow n + m + p$, where

$$h = g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle : m \rightarrow m + p.$$

The subcase that $m = 1$ is called the scalar pairing identity.

- PERMUTATION IDENTITY

$$(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger = \pi \cdot f^\dagger,$$

for all $f : n \rightarrow n + p$ and for all base permutations $\pi : n \rightarrow n$. (Here, π^{-1} denotes the inverse of π .)

Definition 4.1. A *Conway theory* is a preiteration theory satisfying the Conway identities.

It is known that each Conway theory also satisfies the following identities:

- PARAMETER IDENTITY

$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger = f^\dagger \cdot g, \tag{2}$$

all $f : n \rightarrow n + p$, $g : p \rightarrow q$. The particular case that $n = 1$ is called the scalar parameter identity.

- FIXED POINT IDENTITY

$$f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle,$$

for all $f : n \rightarrow n + p$. When $n = 1$, this equation is the scalar fixed point identity.

- COMPOSITION IDENTITY

$$(f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^\dagger = f \cdot \langle (g \cdot \langle f, 0_m \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle,$$

for all $f : n \rightarrow m + p$, $g : m \rightarrow n + p$. When $n = m = 1$, this identity is called the scalar composition identity.

- DOUBLE DAGGER IDENTITY

$$f^{\dagger\dagger} = (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger,$$

for all $f : n \rightarrow n + n + p$. When $n = 1$, this equation is the scalar double dagger identity.

- RIGHT PAIRING IDENTITY

$$\langle f, g \rangle^\dagger = \langle h^\dagger, (g \oplus \rho)^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle \rangle$$

for all $f : n \rightarrow n + m + p$ and $g : m \rightarrow n + m + p$, where

$$\rho = \langle 0_m \oplus \mathbf{1}_n, \mathbf{1}_m \oplus 0_n \rangle \oplus \mathbf{1}_p,$$

$$h = f \cdot \langle \mathbf{1}_n \oplus 0_p, (g \cdot \rho)^\dagger, 0_n \oplus \mathbf{1}_p \rangle.$$

Theorem 4.2. *Each of the following groups of identities is a complete axiomatization of the class of Conway theories:*

1. *The zero, right pairing and permutation identities.*
2. *The parameter, composition and double dagger identities.*
3. *The scalar versions of the parameter, composition, double dagger and pairing identities.*

For proofs and original references, see [11].

Suppose that M is a finite monoid on the set $[n]$, for some $n \geq 1$. For each $i, j \in M$, let us write ij for the product of i and j in the monoid M . In any theory T , we associate with M the base morphisms $\rho_i^M : n \rightarrow n$ defined by

$$j_n \cdot \rho_i^M = (ij)_n$$

for all $i, j \in [n]$. Note that when M is a group, the morphisms ρ_i^M are base permutations. We let τ_n denote the unique base morphism $n \rightarrow 1$.

Definition 4.3 (Ésik [20]). *The monoid-identity associated with the monoid M is the equation*

$$1_n \cdot \langle f \cdot (\rho_1^M \oplus \mathbf{1}_p), \dots, f \cdot (\rho_n^M \oplus \mathbf{1}_p) \rangle^\dagger = (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger, \quad f : 1 \rightarrow n + p.$$

When M is a group, we call the monoid-identity associated with M a *group-identity*.

Definition 4.4. An *iteration theory* is a Conway theory satisfying all of the group-identities.

The axioms of iteration theories give a sound and complete axiomatization of the equational properties of iteration in the following models:

1. Theories of ω -continuous functions on ω -cpo's with a bottom element, or more generally, continuous theories. In these theories, the dagger operation is the least (pre-)fixed point operation.
2. Theories of contraction functions on complete metric spaces, and Elgot's iterative theories. In these theories, the dagger operation is essentially defined by unique fixed points.
3. Theories $\mathbf{Th}_\omega(\mathcal{C})$ of ω -functors on ω -categories \mathcal{C} , or ω -continuous 2-theories. In these theories, the dagger operation is defined by initiality and the iteration theory equations hold up to isomorphism.
4. Matrix theories over completely or countably additive semirings. In these theories the dagger operation is defined by a star operation which involves infinite geometric sums.

For proofs of the above facts, see [11], where original references may be found. In the above theories, except for iterative theories, the dagger operation is constructive in the sense that for each $f : n \rightarrow n + p$, f^\dagger is determined by its “Kleene approximation sequence”.

Remark 4.5. Iteration theories were introduced in [8, 9] and axiomatically in [18]. The axioms in [18] involve the Conway identities and a complicated equation scheme, the commutative identity. In semirings equipped with a star operation, a monoid-identity takes the form of Conway's monoid-identity [16, 26]. Each monoid-identity is an instance of the commutative identity. The completeness of the group-identities in conjunction with the Conway identities is the main result of [21]. This result is a generalization of Krob's axiomatization [26] of the regular sets solving a conjecture of Conway [16]. In fact, it is shown in [21] that the Conway identities and a subcollection of the monoid identities associated with the monoids M_i , $i \in I$ is complete iff each (simple) finite group divides one of the monoids M_i . See [29] for the definition of the divisibility relation. Thus, iteration theories do not have a “finite” axiomatization, see also [19].

5. The main result

Suppose that T is an algebraically complete 2-theory. When $f : n \rightarrow n + p$ and $g : p \rightarrow q$ in T , we define $f_g = f \cdot (\mathbf{1}_n \oplus g) : n \rightarrow n + q$. Note that when (h, u) is an f -algebra, $(h \cdot g, u \cdot g)$ is an f_g -algebra.

We say that the *parameter identity holds in T* , or *T satisfies the parameter identity*, if for any horizontal morphisms f and g , the f_g -algebra $(f^\dagger \cdot g, \mu_f \cdot g)$ is initial, so that Eq. (2) holds up to isomorphism.

Since the dagger operation in algebraically complete categories is defined pointwise, we have:

Proposition 5.1. *If $(\mathcal{C}, \mathcal{F})$ is an algebraically complete category, then $\mathbf{Th}(\mathcal{C}, \mathcal{F})$ satisfies the parameter identity.*

Theorem 5.2. *Suppose that T is an algebraically complete 2-theory satisfying the parameter identity. Then all of the Conway identities hold in T up to isomorphism.*

For a proof, see [11, Theorem 8.4.16]. The fact that the pairing identity holds in all theories $\mathbf{Th}(\mathcal{C}, \mathcal{F})$ where $(\mathcal{C}, \mathcal{F})$ is an algebraically complete category is an extension of a result of Bekič, see [35], and is proved in [31]. See also the product theorem in [23].

Remark 5.3. Theorem 5.2 is a concise statement of several known results that can be found in the papers [7, 22, 23, 31] and in [11]. Nevertheless the proof that the pairing identity, or some other Conway identity holds in an algebraically complete 2-theory satisfying the parameter identity, gives a lot of additional information not covered by Theorem 5.2. See also Lemma 8.4.

Remark 5.4. The fact that Eq. (2) holds up to isomorphism in an algebraically complete 2-theory is a naturality condition of the dagger operation, see [13]. Given an integer n , we can define two functors (in fact 2-functors)

$$T(n, -), T(n, n + -) : T \rightarrow \mathbf{Cat},$$

into the category \mathbf{Cat} of all small categories. Given a morphism $g : p \rightarrow q$, $T(n, g)$ and $T(n, n + g)$ are functors, namely composition with g and $\mathbf{1}_n \oplus g$ on the right, respectively. Then Eq. (2) holds in T iff the following diagram commutes up to isomorphism at the right lower corner, where g is any morphism $p \rightarrow q$:

$$\begin{array}{ccc} T(n, n + p) & \xrightarrow{\quad \dagger_{n,p} \quad} & T(n, p) \\ \downarrow - \cdot (\mathbf{1}_n \oplus g) & & \downarrow - \cdot g \\ T(n, n + q) & \xrightarrow[\quad \dagger_{n,q} \quad]{} & T(n, q) \end{array}$$

Strict commutativity of this diagram would mean that \dagger induces a natural transformation $T(n, n + -) \rightarrow T(n, -)$. The above weak commutativity is the 1-cell level condition for pseudo natural transformations.

The first main result of this paper is the following theorem:

Theorem 5.5. *Any algebraically complete 2-theory satisfying the parameter identity satisfies all of the iteration theory identities up to isomorphism.*

Corollary 5.6. *An equation involving the dagger operation holds up to isomorphism in all algebraically complete 2-theories satisfying the parameter identity iff it holds in all iteration theories.*

Proof. By Theorem 5.5, any such theory satisfies at least the iteration theory identities. But any ω -continuous 2-theory is algebraically complete, and an equation holds in all ω -continuous 2-theories or in all theories $\mathbf{Th}_\omega(\mathcal{C})$ on ω -categories, iff it holds in all iteration theories. See [10, 11]. \square

Corollary 5.7. *An equation involving the dagger operation holds up to isomorphism in all theories $\mathbf{Th}(\mathcal{C}, \mathcal{F})$ where $(\mathcal{C}, \mathcal{F})$ is an algebraically complete category iff it holds in all iteration theories.*

Remark 5.8. There exists an algebraically complete 2-theory not satisfying the parameter identity. See Example 8.4.15 in [11].

Since the equational theory of iteration theories is in the complexity class **P**, we have:

Corollary 5.9. *There is a polynomial time algorithm to decide whether an equation involving dagger holds up to isomorphism in all algebraically complete categories.*

In order to prove Theorem 5.5, by Theorem 5.2 we only need to show that each algebraically complete 2-theory satisfies any group-identity. The details of the argument are included in the next section.

6. Proof of the main result

In this section we give the proof of Theorem 5.5, restated here as Proposition 6.6.

Throughout this section, T denotes a 2-theory, and M denotes a monoid on the set $[n]$. For each $i \in [n]$, we write ρ_i for the base morphism ρ_i^M . Note that

$$\rho_i \cdot \langle \alpha_1, \dots, \alpha_n \rangle = \langle \alpha_{i1}, \dots, \alpha_{in} \rangle,$$

for all 2-cells $\langle \alpha_1, \dots, \alpha_n \rangle : n \rightarrow p$. (As before, we write ij for the product of i and j in the monoid M .) We also note that

$$\rho_i \cdot \rho_j = \rho_{ji},$$

for all $i, j \in [n]$. When $f : 1 \rightarrow n + p$, we define

$$f_M = \langle f \cdot (\rho_1 \oplus \mathbf{1}_p), \dots, f \cdot (\rho_n \oplus \mathbf{1}_p) \rangle : n \rightarrow n + p.$$

Lemma 6.1. *For each $f : 1 \rightarrow n + p$ in T and for each $i \in [n]$,*

$$f_M \cdot (\rho_i \oplus \mathbf{1}_p) = \rho_i \cdot f_M.$$

Proof. We have

$$\begin{aligned}
 j_n \cdot f_M \cdot (\rho_i \oplus \mathbf{1}_p) &= f \cdot (\rho_j \oplus \mathbf{1}_p) \cdot (\rho_i \oplus \mathbf{1}_p) \\
 &= f \cdot (\rho_{ij} \oplus \mathbf{1}_p) \\
 &= (ij)_n \cdot f_M \\
 &= j_n \cdot \rho_i \cdot f_M,
 \end{aligned}$$

for all $i, j \in [n]$. \square

If T is algebraically complete, by Theorem 5.2 all of the Conway identities hold in T . In particular, the permutation identity holds. In fact, we have:

Lemma 6.2. *Suppose that $f: n \rightarrow n + p$, $g: n \rightarrow p$ and $u: f \cdot \langle g, \mathbf{1}_p \rangle \rightarrow g$ in T . Then for any base permutation $\pi: n \rightarrow n$, (g, u) is an initial f -algebra iff $(\pi \cdot g, \pi \cdot u)$ is an initial $(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))$ -algebra.*

The easy proof can be found in [11]. See Proposition 8.4.12.

In the next two lemmas, we assume that M is a group that we prefer to denote by G .

Lemma 6.3. *Suppose that T is algebraically complete and let $f: 1 \rightarrow n + p$ in T . For each $i, j \in [n]$ there is a unique f_G -algebra morphism*

$$\sigma_j^i = \langle \sigma_j^i(1), \dots, \sigma_j^i(n) \rangle : (\rho_i \cdot f_G^\dagger, \rho_i \cdot \mu_{f_G}) \rightarrow (\rho_j \cdot f_G^\dagger, \rho_j \cdot \mu_{f_G}). \quad (3)$$

Moreover, each σ_j^i is an isomorphism and

$$\sigma_j^i \star \sigma_k^j = \sigma_k^i, \quad (4)$$

and

$$\sigma_j^i(k) = \sigma_{ji^{-1}}^1(ik) \quad (5)$$

for all $i, j, k \in [n]$.

Proof. By Lemma 6.1, we have

$$f_G = \rho_i \cdot f_G \cdot (\rho_i^{-1} \oplus \mathbf{1}_p),$$

for each $i \in [n]$. Thus, by Lemma 6.2,

$$\rho_i \cdot (f_G^\dagger, \mu_{f_G}) = (\rho_i \cdot f_G^\dagger, \rho_i \cdot \mu_{f_G})$$

is an initial f_G -algebra. It follows that for each $i, j \in [n]$ there is a unique f_G -algebra morphism (3), and that these morphisms are related by (4). Moreover, each σ_i^i is the identity vertical morphism $\rho_i \cdot f_G^\dagger$, so that each σ_j^i is an isomorphism with inverse σ_i^j .

We now prove (5). Without loss of generality we may assume that the unit element of the group G is the integer 1. Let

$$f_G^\dagger = \langle \bar{f}_1, \dots, \bar{f}_n \rangle, \quad \mu_{f_G} = \langle \mu_1, \dots, \mu_n \rangle.$$

Thus, for each $i \in [n]$,

$$\rho_i \cdot f_G^\dagger = \langle \bar{f}_{i1}, \dots, \bar{f}_{in} \rangle, \quad \rho_i \cdot \mu_{f_G} = \langle \mu_{i1}, \dots, \mu_{in} \rangle.$$

Moreover, for each $i, j \in [n]$, $\sigma_j^i = \langle \sigma_j^i(1), \dots, \sigma_j^i(n) \rangle$ is the unique vertical morphism $\rho_i \cdot f_G^\dagger \rightarrow \rho_j \cdot f_G^\dagger$ such that the square

$$\begin{array}{ccc} f \cdot \langle \bar{f}_{ik1}, \dots, \bar{f}_{ikn}, \mathbf{1}_p \rangle & \xrightarrow{\mu_{ik}} & \bar{f}_{ik} \\ \downarrow f \cdot \langle \sigma_j^i(k1), \dots, \sigma_j^i(kn), \mathbf{1}_p \rangle & & \downarrow \sigma_j^i(k) \\ f \cdot \langle \bar{f}_{jk1}, \dots, \bar{f}_{jkn}, \mathbf{1}_p \rangle & \xrightarrow{\mu_{jk}} & \bar{f}_{jk} \end{array} \quad (6)$$

commutes, for all $k \in [n]$. (This is a restatement of the first part of Lemma 6.3.) In particular,

$$\sigma_{ji}^1 = \langle \sigma_{ji}^1(1), \dots, \sigma_{ji}^1(n) \rangle$$

is the unique vertical morphism $f_G^\dagger \rightarrow \rho_{ji}^1 \cdot f_G^\dagger$ such that the square

$$\begin{array}{ccc} f \cdot \langle \bar{f}_{k1}, \dots, \bar{f}_{kn}, \mathbf{1}_p \rangle & \xrightarrow{\mu_k} & \bar{f}_k \\ \downarrow f \cdot \langle \sigma_{ji}^1(k1), \dots, \sigma_{ji}^1(kn), \mathbf{1}_p \rangle & & \downarrow \sigma_{ji}^1(k) \\ f \cdot \langle \bar{f}_{ji-1k1}, \dots, \bar{f}_{ji-1kn}, \mathbf{1}_p \rangle & \xrightarrow{\mu_{ji-1k}} & \bar{f}_{ji-1k} \end{array} \quad (7)$$

commutes, for all $k \in [n]$. But substituting the product it for k , the squares (7) commute iff so do the squares

$$\begin{array}{ccc} f \cdot \langle \bar{f}_{it1}, \dots, \bar{f}_{itin}, \mathbf{1}_p \rangle & \xrightarrow{\mu_{it}} & \bar{f}_{it} \\ \downarrow f \cdot \langle \sigma_{ji}^1(it1), \dots, \sigma_{ji}^1(itn), \mathbf{1}_p \rangle & & \downarrow \sigma_{ji}^1(it) \\ f \cdot \langle \bar{f}_{jt1}, \dots, \bar{f}_{jtn}, \mathbf{1}_p \rangle & \xrightarrow{\mu_{jt}} & \bar{f}_{jt} \end{array} \quad (8)$$

for all $t \in [n]$. Now (5) follows by comparing (6) and (8). \square

Lemma 6.4. *Let $f : 1 \rightarrow n + p$ in T . If there is an initial f_G -algebra, then there is an initial f_G -algebra whose components are equal.*

Proof. Without loss of generality we may assume that the integer 1 is the unit element of the group G . Using the notations introduced in the proof of Lemma 6.3, define

$$\kappa = (f \cdot \langle \sigma_1^1(1), \dots, \sigma_n^1(1), \mathbf{1}_p \rangle) \star \mu_1,$$

so that κ is the vertical morphism making the following triangle commute:

$$\begin{array}{ccc} f \cdot \langle \bar{f}_1, \dots, \bar{f}_n, \mathbf{1}_p \rangle & \xrightarrow{\mu_1} & \bar{f}_1 \\ \uparrow f \cdot \langle \sigma_1^1(1), \dots, \sigma_n^1(1), \mathbf{1}_p \rangle & \nearrow \kappa & \\ f \cdot \langle \bar{f}_1, \dots, \bar{f}_1, \mathbf{1}_p \rangle & & \end{array}$$

Thus,

$$\tau_n \cdot (\bar{f}_1, \kappa) = (\tau_n \cdot \bar{f}_1, \tau_n \cdot \kappa)$$

is an f_G -algebra whose components are equal. We show that this f_G -algebra is isomorphic to the f_G -algebra (f_G^\dagger, μ_{f_G}) . In fact, we show that the vertical isomorphism

$$\pi = \langle \sigma_1^1(1), \dots, \sigma_n^1(1) \rangle$$

is an f_G -algebra morphism $(\tau_n \cdot \bar{f}_1, \tau_n \cdot \kappa) \rightarrow (f_G^\dagger, \mu_{f_G})$. By the definition of the morphism κ , this follows if the diagram

$$\begin{array}{ccc} f \cdot \langle \bar{f}_1, \dots, \bar{f}_n, \mathbf{1}_p \rangle & \xrightarrow{\mu_1} & \bar{f}_1 \\ \downarrow f \cdot \langle \sigma_1^1(1)^{-1} \star \sigma_{j_1}^1(1), \dots, \sigma_n^1(1)^{-1} \star \sigma_{j_n}^1(1), \mathbf{1}_p \rangle & & \downarrow \sigma_j^1(1) \\ f \cdot \langle \bar{f}_{j_1}, \dots, \bar{f}_{j_n}, \mathbf{1}_p \rangle & \xrightarrow{\mu_j} & \bar{f}_j \end{array} \quad (9)$$

commutes, for all $j \in [n]$. But by Lemma 6.3,

$$\sigma_i^1(1)^{-1} \star \sigma_{ji}^1(i) = \sigma_1^1(1) \star \sigma_{ji}^1(1) = \sigma_{ji}^1(1) = \sigma_j^1(i),$$

for all $i \in [n]$. Thus, (9) commutes for an integer $j \in [n]$ iff so does the square

$$\begin{array}{ccc}
 f \cdot \langle \bar{f}_1, \dots, \bar{f}_n, \mathbf{1}_p \rangle & \xrightarrow{\mu_1} & \bar{f}_1 \\
 \downarrow f \cdot \langle \sigma_j^1(1), \dots, \sigma_j^1(n), \mathbf{1}_p \rangle & & \downarrow \sigma_j^1(1) \\
 f \cdot \langle \bar{f}_{j1}, \dots, \bar{f}_{jn}, \mathbf{1}_p \rangle & \xrightarrow{\mu_j} & \bar{f}_j
 \end{array} \quad (10)$$

But since σ_j^1 is an f_G -algebra morphism $(f_G^\dagger, \mu_{f_G}) \rightarrow \rho_j \cdot \langle f_G^\dagger, \mu_{f_G} \rangle$, the square (10) commutes, by definition. \square

Lemma 6.5. *Suppose that $f: 1 \rightarrow n + p$ in the 2-theory T . If there is an initial f_M -algebra whose components are equal, then any component of this initial f_M -algebra is an initial $f \cdot (\tau_n \oplus \mathbf{1}_p)$ -algebra.*

Proof. Define $g = f \cdot (\tau_n \oplus \mathbf{1}_p)$. Without loss of generality we may assume that the unit element of M is the integer 1. Let

$$\tau_n \cdot (\bar{f}, \bar{\mu}) = (\tau_n \cdot \bar{f}, \tau_n \cdot \bar{\mu})$$

denote an initial f_M -algebra whose components are equal, where $\bar{f}: 1 \rightarrow p$ and $\bar{\mu}: g \cdot \langle \bar{f}, \mathbf{1}_p \rangle \rightarrow \bar{f}$.

If (h, u) is a g -algebra, then, since

$$f_M \cdot \langle \tau_n \cdot h, \mathbf{1}_p \rangle = f_M \cdot (\tau_n \oplus \mathbf{1}_p) \cdot \langle h, \mathbf{1}_p \rangle = \tau_n \cdot g \cdot \langle h, \mathbf{1}_p \rangle,$$

it follows that

$$\tau_n \cdot (h, u) = (\tau_n \cdot h, \tau_n \cdot u)$$

is an f_M -algebra. Hence there exists a unique f_M -algebra morphism

$$v = \langle v_1, \dots, v_n \rangle: \tau_n \cdot (\bar{f}, \bar{\mu}) \rightarrow \tau_n \cdot (h, u).$$

Since v is an f_M -algebra morphism, and since

$$f_M \cdot \langle \tau_n \cdot \bar{f}, \mathbf{1}_p \rangle = \tau_n \cdot f \cdot \langle \tau_n \cdot \bar{f}, \mathbf{1}_p \rangle,$$

the square

$$\begin{array}{ccc}
 \tau_n \cdot f \cdot \langle \tau_n \cdot \bar{f}, \mathbf{1}_p \rangle & \xrightarrow{\tau_n \cdot \bar{\mu}} & \tau_n \cdot \bar{f} \\
 \downarrow f_M \cdot \langle v, \mathbf{1}_p \rangle & & \downarrow v \\
 \tau_n \cdot g \cdot \langle h, \mathbf{1}_p \rangle & \xrightarrow{\tau_n \cdot u} & \tau_n \cdot h
 \end{array} \quad (11)$$

commutes, i.e., for each $k \in [n]$ we have the commutative square

$$\begin{array}{ccc}
 f \cdot \langle \tau_n \cdot \bar{f}, \mathbf{1}_p \rangle & \xrightarrow{\bar{\mu}} & \bar{f} \\
 \downarrow f \cdot \langle v_{k1}, \dots, v_{kn}, \mathbf{1}_p \rangle & & \downarrow v_k \\
 g \cdot \langle h, \mathbf{1}_p \rangle & \xrightarrow{u} & h
 \end{array} \quad (12)$$

Given the integer $i \in [n]$, define

$$v' = \langle v_{i1}, \dots, v_{in} \rangle.$$

The square

$$\begin{array}{ccc}
 \tau_n \cdot f \cdot \langle \tau_n \cdot \bar{f}, \mathbf{1}_p \rangle & \xrightarrow{\tau_n \cdot \bar{\mu}} & \tau_n \cdot \bar{f} \\
 \downarrow f_M \cdot \langle v', \mathbf{1}_p \rangle & & \downarrow v' \\
 \tau_n \cdot g \cdot \langle h, \mathbf{1}_p \rangle & \xrightarrow{\tau_n \cdot u} & \tau_n \cdot h
 \end{array} \quad (13)$$

also commutes. Indeed, for each $j \in [n]$, the j th component of $(\tau_n \cdot \bar{\mu}) \star v'$ is the vertical morphism

$$\bar{\mu} \star v_{ij},$$

and the j th component of the morphism $(f_M \cdot \langle v', \mathbf{1}_p \rangle) \star (\tau_n \cdot h)$ is

$$\begin{aligned}
 (f \cdot (\rho_j \oplus \mathbf{1}_p)) \cdot \langle v', \mathbf{1}_p \rangle \star h &= (f \cdot (\rho_j \oplus \mathbf{1}_p)) \cdot \langle v_{i1}, \dots, v_{in}, \mathbf{1}_p \rangle \star h \\
 &= (f \cdot \langle v_{ij1}, \dots, v_{ijn}, \mathbf{1}_p \rangle) \star h.
 \end{aligned}$$

By (12), these two morphisms are equal.

Comparing (11) and (13), it follows by initiality that $v = v'$, so that $v_1 = v_i$. Since this holds for all $i \in [n]$, we have $v = \tau_n \cdot z$, for the vertical morphism $z = v_1 : \bar{f} \rightarrow h$. Thus, when $k = 1$, the commutative square (12) may be redrawn as

$$\begin{array}{ccc}
 f \cdot \langle \tau_n \cdot \bar{f}, \mathbf{1}_p \rangle & \xrightarrow{\bar{\mu}} & \bar{f} \\
 \downarrow f \cdot \langle \tau_n \cdot z, \mathbf{1}_p \rangle & & \downarrow z \\
 g \cdot \langle h, \mathbf{1}_p \rangle & \xrightarrow{u} & h
 \end{array}$$

Since

$$f \cdot \langle \tau_n \cdot \bar{f}, \mathbf{1}_p \rangle = g \cdot \langle \bar{f}, \mathbf{1}_p \rangle, \quad f \cdot \langle \tau_n \cdot z, \mathbf{1}_p \rangle = g \cdot \langle z, \mathbf{1}_p \rangle,$$

this means that z is a g -algebra morphism $(\bar{f}, \bar{\mu}) \rightarrow (h, u)$. If z' is also a g -algebra morphism $(\bar{f}, \bar{\mu}) \rightarrow (h, u)$, then $\tau_n \cdot z'$ is an f_M -algebra morphism $\tau_n \cdot (\bar{f}, \bar{\mu}) \rightarrow \tau_n \cdot (h, u)$. Since $\tau_n \cdot (\bar{f}, \bar{\mu})$ is initial, we have $\tau_n \cdot z' = \tau_n \cdot z$, so that $z' = z$. \square

Proposition 6.6. *Suppose that T is an algebraically complete 2-theory. Then the dagger operation on T satisfies each group-identity up to isomorphism.*

Proof. This follows from Lemmas 6.4 and 6.5. \square

7. Iterating vertical morphisms

It is shown in [11], see Subsection 1.4.1 and Section 3.7, that when T is a 2-theory, then so is the 2-category \mathbf{Cell}_T , whose horizontal morphisms are the 2-cells of T . In \mathbf{Cell}_T , a horizontal morphism $n \rightarrow p$ is a 2-cell $(u: f \rightarrow g): n \rightarrow p$ that in this section we will denote (f, g, u) for typographical reasons. And if (f, g, u) and (f', g', u') are horizontal morphisms $n \rightarrow p$ in \mathbf{Cell}_T , a vertical morphism

$$(f, g, u) \rightarrow (f', g', u')$$

is a pair (v_1, v_2) consisting of vertical morphisms $v_1: f \rightarrow f'$ and $v_2: g \rightarrow g'$ such that

$$u \star v_2 = v_1 \star u'.$$

Horizontal composition of horizontal \mathbf{Cell}_T -morphisms as well as horizontal identities have already been defined. Thus, if

$$(f, g, u): n \rightarrow p \quad \text{and} \quad (f', g', u'): p \rightarrow q,$$

then

$$(f, g, u) \cdot (f', g', u') = (f \cdot f', g \cdot g', u \cdot u'): n \rightarrow q.$$

As for vertical composition, suppose that

$$(f, g, u), (f', g', u') \quad \text{and} \quad (f'', g'', u'')$$

are horizontal morphisms $n \rightarrow p$ and that

$$(v_1, v_2): (f, g, u) \rightarrow (f', g', u'), \quad (v'_1, v'_2): (f', g', u') \rightarrow (f'', g'', u'')$$

are vertical morphisms in \mathbf{Cell}_T . The vertical composite $(v_1, v_2) \star (v'_1, v'_2)$ is defined to be the morphism

$$(v_1 \star v'_1, v_2 \star v'_2): (f, g, u) \rightarrow (f'', g'', u'').$$

For each horizontal morphism $(f, g, u): n \rightarrow p$, the vertical identity $(f, g, u) \rightarrow (f, g, u)$ is the ordered pair (f, g) consisting of two vertical T -identity morphisms. To define horizontal composition of vertical morphisms, assume that

$$(v_1, v_2): (f, g, u) \rightarrow (f', g', u'), \quad (w_1, w_2): (h, k, z) \rightarrow (h', k', z')$$

in \mathbf{Cell}_T , where

$$(f, g, u), (f', g', u'): n \rightarrow p, \quad (h, k, z), (h', k', z'): p \rightarrow q.$$

Then we define:

$$(v_1, v_2) \cdot (w_1, w_2) = (v_1 \cdot w_1, v_2 \cdot w_2).$$

Lastly, for each $i \in [n]$, $n \geq 0$, the i th distinguished morphism $1 \rightarrow n$ in \mathbf{Cell}_T is (i_n, i_n, i_n) . We omit the details of the straightforward calculation that \mathbf{Cell}_T is a 2-theory whenever T is. We only note that for $(f, g, u): n \rightarrow p$ and $(f', g', u'): m \rightarrow p$, the pairing

$$\langle (f, g, u), (f', g', u') \rangle$$

is

$$\langle \langle f, f' \rangle, \langle g, g' \rangle, \langle u, u' \rangle \rangle: n + m \rightarrow p,$$

and if

$$(h, k, z): n \rightarrow p \quad \text{and} \quad (h', k', z'): m \rightarrow p$$

and

$$(v_1, v_2): (f, g, u) \rightarrow (h, k, z), \quad (w_1, w_2): (f', g', u') \rightarrow (h', k', z'),$$

then

$$\langle (v_1, v_2), (w_1, w_2) \rangle = \langle (v_1, w_1), (v_2, w_2) \rangle.$$

The separated sum of two horizontal or vertical morphisms has a similar description.

Suppose that $f: n \rightarrow n + p$ and $g: n \rightarrow n + p$ in a 2-theory T such that the initial algebras (f^\dagger, μ_f) and (g^\dagger, μ_g) exist. Suppose further that u is a vertical morphism $f \rightarrow g$, giving an interpretation of the “data type constructor” f in g . By the next lemma, it is possible to give a canonical interpretation of f^\dagger in g^\dagger .

Lemma 7.1. *If (f^\dagger, μ_f) is an initial f -algebra and (\bar{g}, v) is a g -algebra, and if $u: f \rightarrow g$, then there is a unique vertical morphism $\bar{u}: f^\dagger \rightarrow \bar{g}$ such that the following*

square commutes:

$$\begin{array}{ccc}
 f \cdot \langle f^\dagger, \mathbf{1}_p \rangle & \xrightarrow{\mu_f} & f^\dagger \\
 u \cdot \langle \bar{u}, \mathbf{1}_p \rangle \downarrow & & \downarrow \bar{u} \\
 g \cdot \langle \bar{g}, \mathbf{1}_p \rangle & \xrightarrow{v} & \bar{g}
 \end{array} \quad (14)$$

Proof. Clearly, (14) commutes iff so does the following square:

$$\begin{array}{ccc}
 f \cdot \langle f^\dagger, \mathbf{1}_p \rangle & \xrightarrow{\mu_f} & f^\dagger \\
 f \cdot \langle \bar{u}, \mathbf{1}_p \rangle \downarrow & & \downarrow \bar{u} \\
 f \cdot \langle \bar{g}, \mathbf{1}_p \rangle & \xrightarrow{(u \cdot \langle \bar{g}, \mathbf{1}_p \rangle) \star v} & \bar{g}
 \end{array}$$

Since (f^\dagger, μ_f) is an initial f -algebra, \bar{u} exists and is unique. \square

Proposition 7.2. Suppose that T is an algebraically complete 2-theory with a specified initial f -algebra (f^\dagger, μ_f) , for each $f : n \rightarrow n + p$. Then for any integers $n, p \geq 0$, the map

$$f : n \rightarrow n + p \mapsto f^\dagger : n \rightarrow p$$

is the object map of a unique functor

$$^\dagger : T(n, n + p) \rightarrow T(n, p)$$

such that $\mu = (\mu_f)$, $f : n \rightarrow n + p$ is a natural transformation (in fact isomorphism)

$$(-) \cdot \langle (-)^\dagger, \mathbf{1}_p \rangle \rightarrow (-)^\dagger.$$

Proof. Taking $(\bar{g}, v) = (g^\dagger, \mu_g)$ in Lemma 7.1, it follows that there is a unique $u^\dagger : f^\dagger \rightarrow g^\dagger$ such that

$$\mu_f \star u^\dagger = (u \cdot \langle u^\dagger, \mathbf{1}_p \rangle) \star \mu_g. \quad (15)$$

It follows that for each $n, p \geq 0$, † is a functor $T(n, n + p) \rightarrow T(n, p)$. Moreover, (μ_f) is natural in f . This uniqueness of the functor † is immediate from Lemma 7.1. \square

This defines the dagger operation $u \mapsto u^\dagger$ on vertical morphisms in any algebraically complete 2-theory. Below we will show that this operation also satisfies the iteration theory identities. In fact, we reduce the proof of this fact to Theorem 5.5.

Suppose that T is a 2-theory and

$$(f, g, u) : n \rightarrow n + p$$

in \mathbf{Cell}_T . An (f, g, u) -algebra consists of a horizontal morphism $(\bar{f}, \bar{g}, \bar{u}) : n \rightarrow p$ and a vertical morphism

$$(v_1, v_2) : (f, g, u) \cdot \langle (\bar{f}, \bar{g}, \bar{u}), \mathbf{1}_p \rangle \rightarrow (\bar{f}, \bar{g}, \bar{u})$$

in \mathbf{Cell}_T , so that

$$v_1 : f \cdot \langle \bar{f}, \mathbf{1}_p \rangle \rightarrow \bar{f}, \quad v_2 : g \cdot \langle \bar{g}, \mathbf{1}_p \rangle \rightarrow \bar{g}$$

in T . Since (v_1, v_2) is a vertical morphism in \mathbf{Cell}_T , the following square commutes:

$$\begin{array}{ccc} f \cdot \langle \bar{f}, \mathbf{1}_p \rangle & \xrightarrow{v_1} & \bar{f} \\ u \cdot \langle \bar{u}, \mathbf{1}_p \rangle \downarrow & & \downarrow \bar{u} \\ g \cdot \langle \bar{g}, \mathbf{1}_p \rangle & \xrightarrow{v_2} & \bar{g} \end{array}$$

If $((f', g', u'), (v'_1, v'_2))$ is another (f, g) -algebra, an (f, g, u) -algebra morphism

$$((\bar{f}, \bar{g}, \bar{u}), (v_1, v_2)) \rightarrow ((f', g', u'), (v'_1, v'_2))$$

is a vertical \mathbf{Cell}_T -morphism

$$(w_1, w_2) : (\bar{f}, \bar{g}, \bar{u}) \rightarrow (f', g', u')$$

such that the following two squares also commute:

$$\begin{array}{ccc} \bar{f} & \xrightarrow{\bar{u}} & \bar{g} \\ w_1 \downarrow & & \downarrow w_2 \\ f' & \xrightarrow{u'} & g' \end{array} \quad (16)$$

$$\begin{array}{ccc} f \cdot \langle \bar{f}, \mathbf{1}_p \rangle & \xrightarrow{u \cdot \langle \bar{u}, \mathbf{1}_p \rangle} & g \cdot \langle \bar{g}, \mathbf{1}_p \rangle \\ f \cdot \langle w_1, \mathbf{1}_p \rangle \downarrow & & \downarrow g \cdot \langle w_2, \mathbf{1}_p \rangle \\ f \cdot \langle f', \mathbf{1}_p \rangle & \xrightarrow{u \cdot \langle u', \mathbf{1}_p \rangle} & g \cdot \langle g', \mathbf{1}_p \rangle \end{array} \quad (17)$$

Moreover, w_1 is an f -algebra morphism $(\bar{f}, v_1) \rightarrow (f', v'_1)$ and w_2 is a g -algebra morphism $(\bar{g}, v_2) \rightarrow (g', v'_2)$.

However, note that (17) commutes whenever (16) does.

Theorem 7.3. *If T is an algebraically complete 2-theory, then \mathbf{Cell}_T is also algebraically complete. Moreover, if the parameter identity holds in T , then it holds in \mathbf{Cell}_T .*

Proof. We need to show that each 2-cell (i.e., horizontal \mathbf{Cell}_T -morphism)

$$(f, g, u) : n \rightarrow n + p$$

has an initial algebra. But let (f^\dagger, μ_f) be an initial f -algebra and (g^\dagger, μ_g) an initial g -algebra. By Lemma 7.1, there is a unique vertical T -morphism $u^\dagger : f^\dagger \rightarrow g^\dagger$ such that (15) holds. This defines the (f, g, u) -algebra

$$((f^\dagger, g^\dagger, u^\dagger), (\mu_f, \mu_g)).$$

We claim that this is initial.

Suppose that $((\bar{f}, \bar{g}, \bar{u}), (v_f, v_g))$ is also an (f, g, u) -algebra, so that the following square commutes:

$$\begin{array}{ccc} f \cdot \langle \bar{f}, \mathbf{1}_p \rangle & \xrightarrow{v_f} & \bar{f} \\ u \cdot \langle \bar{u}, \mathbf{1}_p \rangle \downarrow & & \downarrow \bar{u} \\ g \cdot \langle \bar{g}, \mathbf{1}_p \rangle & \xrightarrow{v_g} & \bar{g} \end{array} \quad (18)$$

We need to find vertical T -morphisms

$$w_f : f^\dagger \rightarrow \bar{f} \quad \text{and} \quad w_g : g^\dagger \rightarrow \bar{g}$$

such that (w_f, w_g) is an (f, g, u) -algebra morphism

$$((f^\dagger, g^\dagger, u^\dagger), (\mu_f, \mu_g)) \rightarrow ((\bar{f}, \bar{g}, \bar{u}), (v_f, v_g)),$$

i.e., such that w_f is an f -algebra morphism $(f^\dagger, \mu_f) \rightarrow (\bar{f}, v_f)$, w_g is a g -algebra morphism $(g^\dagger, \mu_g) \rightarrow (\bar{g}, v_g)$, and

$$u^\dagger \star w_g = w_f \star \bar{u}, \quad (19)$$

$$(u \cdot \langle u^\dagger, \mathbf{1}_p \rangle) \star (g \cdot \langle w_g, \mathbf{1}_p \rangle) = (f \cdot \langle w_f, \mathbf{1}_p \rangle) \star (u \cdot \langle \bar{u}, \mathbf{1}_p \rangle). \quad (20)$$

By initiality, w_f and w_g are unique. Moreover, since the left-hand side of (20) is $u \cdot \langle u^\dagger \star w_g, \mathbf{1}_p \rangle$ and the right-hand side is $u \cdot \langle w_f \star \bar{u}, \mathbf{1}_p \rangle$, (19) implies (20). To prove

(19), note that

$$\begin{aligned}\mu_f \star u^\dagger \star w_g &= (u \cdot \langle u^\dagger, \mathbf{1}_p \rangle) \star \mu_g \star w_g = (u \cdot \langle u^\dagger, \mathbf{1}_p \rangle) \star (g \cdot \langle w_g, \mathbf{1}_p \rangle) \star v_g \\ &= (u \cdot \langle u^\dagger \star w_g, \mathbf{1}_p \rangle) \star v_g,\end{aligned}$$

using (15) and the fact that w_g is a g -algebra morphism. Also,

$$\begin{aligned}\mu_f \star w_f \star \bar{u} &= (f \cdot \langle w_f, \mathbf{1}_p \rangle) \star v_f \star \bar{u} = (f \cdot \langle w_f, \mathbf{1}_p \rangle) \star (u \cdot \langle \bar{u}, \mathbf{1}_p \rangle) \star v_g \\ &= (u \cdot \langle w_f \star \bar{u}, \mathbf{1}_p \rangle) \star v_g,\end{aligned}$$

since w_f is an f -algebra morphism, and by (18). Eq. (19) now follows from Lemma 7.1.

Suppose now that the parameter identity holds in T , so that for each $f: n \rightarrow n+p$ and $g: p \rightarrow q$,

$$(f^\dagger \cdot g, \mu_f \cdot g)$$

is an initial f_g -algebra, where f_g denotes the morphism $f \cdot (\mathbf{1}_n \oplus g): n \rightarrow n+q$.

To prove that the parameter identity also holds in \mathbf{Cell}_T , suppose that $F = (f, g, u): n \rightarrow n+p$ and $G = (h, k, v): p \rightarrow q$ in \mathbf{Cell}_T . We already know that $(F^\dagger, \mu_F) = ((f^\dagger, g^\dagger, u^\dagger), (\mu_f, \mu_g))$ is an initial F -algebra. We need to show that

$$(F^\dagger \cdot G, \mu_F \cdot G) = ((f^\dagger \cdot h, g^\dagger \cdot k, u^\dagger \cdot v), (\mu_f \cdot h, \mu_g \cdot k)) \quad (21)$$

is initial for

$$F_G = (f, g, u) \cdot (\mathbf{1}_n \oplus (h, k, v)) = (f \cdot (\mathbf{1}_n \oplus h), g \cdot (\mathbf{1}_n \oplus k), u \cdot (\mathbf{1}_n \oplus v)).$$

Note that (21) defines an F_G -algebra. Let $(\bar{F}, w) = ((\bar{f}, \bar{g}, \bar{u}), (w_f, w_g))$ be an F_G -algebra, so that the following square commutes:

$$\begin{array}{ccc} f \cdot \langle \bar{f}, h \rangle & \xrightarrow{w_f} & \bar{f} \\ u \cdot \langle \bar{u}, v \rangle \downarrow & & \downarrow \bar{u} \\ g \cdot \langle \bar{g}, k \rangle & \xrightarrow{w_g} & \bar{g} \end{array} \quad (22)$$

Since (\bar{f}, w_f) is an f_h -algebra, and since the parameter identity holds in T , there is a unique f_h -algebra morphism

$$z_f: (f^\dagger \cdot h, \mu_f \cdot h) \rightarrow (\bar{f}, w_f).$$

Similarly, there is a unique g_k -algebra morphism

$$z_g: (g^\dagger \cdot k, \mu_g \cdot k) \rightarrow (\bar{g}, w_g).$$

Thus, the following two squares also commute:

$$\begin{array}{ccc}
 f \cdot \langle f^\dagger \cdot h, h \rangle & \xrightarrow{\mu_f \cdot h} & f^\dagger \cdot h \\
 \downarrow f \cdot \langle z_f, h \rangle & & \downarrow z_f \\
 f \cdot \langle \bar{f}, h \rangle & \xrightarrow{w_f} & \bar{f}
 \end{array} \quad (23)$$

$$\begin{array}{ccc}
 g \cdot \langle g^\dagger \cdot k, k \rangle & \xrightarrow{\mu_g \cdot k} & g^\dagger \cdot k \\
 \downarrow g \cdot \langle z_g, k \rangle & & \downarrow z_g \\
 g \cdot \langle \bar{g}, k \rangle & \xrightarrow{w_g} & \bar{g}
 \end{array} \quad (24)$$

To conclude that (z_f, z_g) is an F_G -algebra morphism $(F^\dagger \cdot G, \mu_F \cdot G) \rightarrow (\bar{F}, w)$, we must prove that

$$z_f \star \bar{u} = (u^\dagger \cdot v) \star z_g. \quad (25)$$

But

$$\begin{aligned}
 (\mu_f \cdot h) \star z_f \star \bar{u} &= (f \cdot \langle z_f, h \rangle) \star w_f \star \bar{u} \\
 &= (f \cdot \langle z_f, h \rangle) \star (u \cdot \langle \bar{u}, v \rangle) \star w_g \\
 &= (u \cdot \langle z_f \star \bar{u}, v \rangle) \star w_g \\
 &= (u \cdot (\mathbf{1}_n \oplus v)) \star \langle z_f \star \bar{u}, \mathbf{1}_q \rangle \star w_g,
 \end{aligned}$$

by (23) and (22), and

$$\begin{aligned}
 (\mu_f \cdot h) \star (u^\dagger \cdot v) \star z_g &= (u \cdot \langle u^\dagger \cdot v, v \rangle) \star (\mu_g \cdot k) \star z_g \\
 &= (u \cdot \langle u^\dagger \cdot v, v \rangle) \star (g \cdot \langle z_g, k \rangle) \star w_g \\
 &= (u \cdot \langle (u^\dagger \cdot v) \star z_g, v \rangle) \star w_g \\
 &= (u \cdot (\mathbf{1}_n \oplus v)) \star \langle (u^\dagger \cdot v) \star z_g, \mathbf{1}_q \rangle \star w_g,
 \end{aligned}$$

by (24) and since $(F^\dagger \cdot G, \mu_F \cdot G)$ is an F_G -algebra. Now (25) follows by applying Lemma 7.1 to the morphisms $f \cdot (\mathbf{1}_n \oplus h)$, $g \cdot (\mathbf{1}_n \oplus k)$ and $u \cdot (\mathbf{1}_n \oplus v)$. The uniqueness of the F_G -algebra morphism (z_f, z_g) is immediate, since $(f^\dagger \cdot h, \mu_f \cdot h)$ and $(g^\dagger \cdot k, \mu_g \cdot k)$ are initial. \square

Corollary 7.4. *Suppose that T is an algebraically complete 2-theory satisfying the parameter identity. Then all of the iteration theory identities hold in \mathbf{Cell}_T up to isomorphism.*

Remark 7.5. Suppose that T is an algebraically complete 2-theory. When $f : n \rightarrow n + p$ in T , define

$$f^\nabla = (f \cdot \langle f^\dagger, \mathbf{1}_p \rangle, f^\dagger, \mu_f),$$

so that f^∇ is a 2-cell and hence a morphism $n \rightarrow p$ in \mathbf{Cell}_T . If also $g : n \rightarrow n + p$ and $u : f \rightarrow g$ in T , let

$$u^\nabla = (u \cdot \langle u^\dagger, \mathbf{1}_p \rangle, u^\dagger).$$

Then ∇ is a functor $T(n, n + p) \rightarrow \mathbf{Cell}_T(n, p)$.

Remark 7.6. The assumption made in Section 5 and Section 7 that for each $f : n \rightarrow n + p$ and $g : p \rightarrow q$, $(f^\dagger \cdot g, \mu_g \cdot g)$ is an initial f_g -algebra can be expressed as a naturality condition for the functor ∇ .

Given an integer n , we can define two functors

$$\mathbf{Cell}_T(n, -), \mathbf{Cell}_T(n, n + -) : T \rightarrow \mathbf{Cat},$$

as in Remark 5.4. Then the naturality condition is that the following square commutes up to isomorphism for any $g : n \rightarrow p$ in T :

$$\begin{array}{ccc} \mathbf{Cell}_T(n, n + p) & \xrightarrow{\nabla_{n,p}} & \mathbf{Cell}_T(n, p) \\ \downarrow - \cdot (\mathbf{1}_n \oplus g) & & \downarrow - \cdot g \\ \mathbf{Cell}_T(n, n + q) & \xrightarrow{\nabla_{n,q}} & \mathbf{Cell}_T(n, q) \end{array}$$

where $\nabla_{n,p}$ is the restriction of ∇ to $\mathbf{Cell}_T(n, n + p)$.

8. An extension of the main result

The concept of an algebraically complete category or 2-theory is based on initial algebras which, in the ordered setting, correspond to least pre-fixed points. But sometimes least fixed points exist when least pre-fixed points do not. Our proof that the dagger operation on the horizontal morphisms of an algebraically complete category satisfies the group-identities also works in the case that dagger is defined by weak initial algebras that correspond to least fixed points.

Definition 8.1. Suppose that T is a 2-theory and $f : n \rightarrow n + p$ in T . A *weak initial f -algebra* is an f -algebra (g, u) , where u is an isomorphism, such that for each f -algebra (h, v) with v an isomorphism there exists a unique f -algebra morphism $(g, u) \rightarrow (h, v)$.

Definition 8.2. A *weak algebraically complete 2-theory* is a 2-theory T together with a specified weak initial algebra (f^\dagger, μ_f) , for each horizontal morphism $f: n \rightarrow n+p$.

Thus, any algebraically complete 2-theory is a weak algebraically complete 2-theory. (Note that our terminology is not standard, since a weak initial object in a category \mathcal{C} is an object C_0 such that for any \mathcal{C} -object C there is at most one morphism $C_0 \rightarrow C$.)

Example 8.3. Suppose that \mathcal{D} is a category and G is an endofunctor $\mathcal{D} \rightarrow \mathcal{D}$. Freyd [22, 23] calls a G -algebra (D_0, f_0) *invariant* if f_0 is an isomorphism. If in addition there is a unique G -algebra morphism $(D_0, f_0) \rightarrow (D, f)$, for each invariant G -algebra (D, f) , then (D_0, f_0) is an initial invariant G -algebra, or in our terminology, a weak initial G -algebra.

Suppose now that \mathcal{C} is a category and \mathcal{F} is a collection of functors $\mathcal{C}^p \rightarrow \mathcal{C}^n$, $n, p \geq 0$, closed under composition and tupling and containing the projections, so that \mathcal{F} determines the 2-theory $\mathbf{Th}(\mathcal{C}, \mathcal{F})$. Suppose that for each $F: \mathcal{C}^{n+p} \rightarrow \mathcal{C}^n$ in $\mathbf{Th}(\mathcal{C}, \mathcal{F})$ and \mathcal{C}^p -object y , the functor $F_y: \mathcal{C}^n \rightarrow \mathcal{C}^n$ has a weak initial algebra $(F_y^\dagger, \mu_{F,y})$. Again, the assignment $y \rightarrow F_y^\dagger y$ may be extended to a functor $F^\dagger: \mathcal{C}^p \rightarrow \mathcal{C}^n$. If F^\dagger is in \mathcal{F} whenever F is, then $\mathbf{Th}(\mathcal{C}, \mathcal{F})$ is a weak algebraically complete 2-theory that we will call a *weak algebraically complete category*. A weak algebraically complete category is not necessarily an algebraically complete category. See [20] for examples involving the ordered case.

Lemma 8.4. Suppose that $f: n \rightarrow m+p$ and $g: m \rightarrow n+p$ in a 2-theory T . Then there is a (weak) initial $(f \cdot \langle g, 0_n \oplus 1_p \rangle)$ -algebra iff there is a (weak) initial $(g \cdot \langle f, 0_m \oplus 1_p \rangle)$ -algebra. Moreover, if (h, u) is a (weak) initial $(f \cdot \langle g, 0_n \oplus 1_p \rangle)$ -algebra, then

$$g \cdot \langle (h, u), 1_p \rangle = (g \cdot \langle h, 1_p \rangle, g \cdot \langle u, 1_p \rangle)$$

is a (weak) initial $(g \cdot \langle f, 0_m \oplus 1_p \rangle)$ -algebra.

Proof. If (h, u) is an $(f \cdot \langle g, 0_n \oplus 1_p \rangle)$ -algebra, then $g \cdot \langle (h, u), 1_p \rangle$ is a $(g \cdot \langle f, 0_m \oplus 1_p \rangle)$ -algebra. And if u is an isomorphism, then so is the morphism $g \cdot \langle u, 1_p \rangle$. This correspondence may be extended to morphisms of the algebras, since if v is an $(f \cdot \langle g, 0_n \oplus 1_p \rangle)$ -algebra morphism $(h, u) \rightarrow (h', u')$, then $g \cdot \langle v, 1_p \rangle$ is a $(g \cdot \langle f, 0_m \oplus 1_p \rangle)$ -algebra morphism $g \cdot \langle (h, u), 1_p \rangle \rightarrow g \cdot \langle (h', u'), 1_p \rangle$. In the same way, we may assign to any $(g \cdot \langle f, 0_m \oplus 1_p \rangle)$ -algebra morphism an $(f \cdot \langle g, 0_n \oplus 1_p \rangle)$ -algebra morphism. It follows that these assignments preserve (weak) initial algebras. \square

Corollary 8.5. The composition identity holds in any weak algebraically complete 2-theory up to isomorphism.

Below we will say that a weak algebraically complete 2-theory satisfies the parameter identity if for any $f: n \rightarrow n+p$ and $g: p \rightarrow q$, if (f^\dagger, μ_f) is a weak initial f -algebra,

then $(f^\dagger \cdot g, \mu_f \cdot g)$ is a weak initial f_g -algebra. Note that any weak algebraically complete category satisfies the parameter identity.

Theorem 8.6. *Suppose that T is a weak algebraically complete 2-theory satisfying the parameter identity. If the double dagger identity holds in T up to isomorphism, then so do all of the iteration theory identities.*

Proof. By Corollary 8.5 and Theorem 4.2, all of the Conway theory identities hold in T . The argument that the group-identities hold is the same as for algebraically complete 2-theories. \square

Corollary 8.7. *An identity involving dagger holds up to isomorphism in all weak algebraically complete 2-theories satisfying the parameter identity and the double dagger identity iff it holds in all iteration theories.*

However, in weak algebraically complete 2-theories, it is not possible to define an induced canonical dagger operation on the vertical morphisms other than the isomorphisms. In fact, in [20], an example of an ordered weak algebraically complete 2-theory is given, so that there is at most one vertical morphism between any two horizontal morphisms, such that the dagger operation on the horizontal morphisms is not monotonic.

9. An example

We use equational reasoning to show that two data type specifications are equivalent. We will make use of the scalar commutative identity [11] which holds in all iteration theories and which may be seen as a generalization of the monoid identities:

$$(f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger = \langle f \cdot (\rho_1 \oplus \mathbf{1}_p), \dots, f \cdot (\rho_n \oplus \mathbf{1}_p) \rangle^\dagger,$$

for all $f: 1 \rightarrow n + p$ and for all base morphisms $\rho_i: n \rightarrow n$, $i \in [n]$.

Let **Set** denote the ω -category of sets, and let F and G denote the following functors, specified by their object maps:

$$\begin{aligned} F: \mathbf{Set} &\rightarrow \mathbf{Set}, \\ X &\mapsto A + X \times X \times X \end{aligned}$$

and

$$\begin{aligned} G: \mathbf{Set}^3 &\rightarrow \mathbf{Set}, \\ (X, Y, Z) &\mapsto A + X \times Y \times Z, \end{aligned}$$

where A is a given set. Thus F defines the data type F^\dagger of all ternary trees with leaves labeled in the set A .

Let H denote the functor $\mathbf{Set}^2 \rightarrow \mathbf{Set}$:

$$\begin{aligned} H &= G \cdot \langle 1_2, 2_2, G^\dagger \cdot \langle 2_2, 1_2 \rangle \rangle, \\ (X, Y) &\mapsto G(X, Y, G^\dagger(Y, X)) \\ &= A + X \times Y \times G^\dagger(Y, X). \end{aligned}$$

Moreover, define

$$\begin{aligned} K &= G \cdot \langle 1_1, H^\dagger, G^\dagger \cdot \langle 1_1, H^\dagger \rangle \rangle, \\ X &\mapsto G(X, H^\dagger(X), G^\dagger(X, H^\dagger(X))) \\ &= A + X \times H^\dagger(X) \times G^\dagger(X, H^\dagger(X)), \end{aligned}$$

so that K is a functor $\mathbf{Set} \rightarrow \mathbf{Set}$. In fact, all of the functors F, G, H, K are ω -functors. It is not immediately clear what data type the functor K defines. But using the right pairing identity twice, we have

$$K^\dagger = 1_3 \cdot \langle G, G \cdot \langle 2_3, 1_3, 3_3 \rangle, G \cdot \langle 3_3, 1_3, 2_3 \rangle \rangle^\dagger,$$

i.e., K^\dagger is the first component of the initial solution of the system of equations:

$$X = G(X, Y, Z), \quad Y = G(Y, X, Z), \quad Z = G(Z, X, Y).$$

By the scalar commutative identity, K^\dagger is isomorphic to F^\dagger , so that K also defines the data type of ternary trees with leaves labeled in A .

10. Discussion and further results

The main result was that the iteration theory identities hold in all algebraically complete 2-theories satisfying the parameter identity. Our argument was based on the completeness of the Conway identities and the group-identities for iteration theories. But the proof that the pairing identity holds, say, shows that the two sides of an instance of the pairing identity are not simply isomorphic but *canonically isomorphic*. Similarly, when $f : 1 \rightarrow n + p$ in an algebraically complete 2-theory T and G is a group on the set $[n]$, and if $g = f \cdot (\tau_n \oplus 1_p)$, then f_G^\dagger and g^\dagger are not only isomorphic, but the unique f_G -algebra morphism $(f_G^\dagger, \mu_{f_G}) \rightarrow \tau_n \cdot (g^\dagger, \mu_g)$ is an isomorphism $f_G^\dagger \rightarrow \tau_n \cdot g^\dagger$. We conjecture that a canonical isomorphism may be established in connection with any iteration theory identity.

The dagger operation on algebraically complete 2-theories satisfying the parameter identity interacts with the 2-categorical and cartesian structure in a smooth way. The properties of this interaction are captured by the axioms of *iteration 2-theories* studied in [14]. In that paper, the authors give a concrete description of the free iteration 2-theory generated by a “2-signature” Σ . The description generalizes the representation of the free iteration theories [11] by the regular trees. It is conjectured that the free algebraically complete 2-theories satisfying the parameter identity have a related concrete description.

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